New constant and trigonometric 4*4 solutions to the Yang-Baxter equations

1992 J. Phys. A: Math. Gen. 25 L63

(http://iopscience.iop.org/0305-4470/25/2/006)

J. Phys. A: Math. Gen. 25 (1992) L63–L68. Printed in the UK

LETTER TO THE EDITOR

New constant and trigonometric 4×4 solutions to the Yang–Baxter equations

Ladislav Hlavatý

Institute of Physics, Czechoslovak Academy of Sciences, Na Slovance 2, 180 40 Prague 8, Czechoslovakia

Received 23 September 1991

Abstract. A classification of triangular constant solutions of the Yang–Baxter equations is performed. The formulae for the Baxterization of braid group representations are used for finding trigonometric solutions of the Yang–Baxter equations from the constant ones. Some of the obtained solutions correspond to known solutions but several new ones are obtained as well. The calculations confirm that the Baxterization formulae for more than two eigenvalues of the $\hat{R}$-matrix are not universal.

The Yang–Baxter equations (YBE) have appeared in many branches of theoretical physics and recently also in mathematics [1]. Therefore their solutions have a rather wide field of applications.

A few years ago the author of this letter published a classification of 4×4 matrices of the eight-or-less vertex form

$$R=\begin{pmatrix} q & 0 & 0 & d \\ 0 & r & c & 0 \\ 0 & b & s & 0 \\ a & 0 & 0 & t \end{pmatrix} \qquad \det R \neq 0 \tag{1}$$

that satisfy the Yang–Baxter equations

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12} \tag{2}$$

and their 'spectral-dependent' counterparts [2]. For the reader's convenience and as they will be used in the following, the eight-or-less vertex constant solutions are listed in the appendix.

In this letter we shall perform the classification of triangular constant solutions. The classification of general 4×4 solutions seems to be a rather complicated problem.

Recently, formulae for the trigonometric Yang–Baxterization of the constant R-matrices that satisfy the YBE (or equivalently braid groups relations) were found [3]. We will apply the formulae for Baxterizing the constant solutions.

In [2] the spectral-dependent solutions $R(u,v)$ were obtained by the requirement that $R(u,u)$ be equal to a given constant solution R for some u. Most of the solutions were genuine functions of two variables (cf [4]), not only of their difference or ratio or any other combination. Obviously the spectral-dependent solutions that we shall get by the Baxterization formulae will differ from those obtained in [2] because the trigonometric solutions are of the form $R(u,v)=R(u/v)$ and the initial condition $R(0)=R$ used in [3] cannot be realized as $R(u,u)=R$.

0305-4470/92/020063+06$04.50

An attempt to classify the eight-or-less vertex solutions dependent on the difference $u-v$ of spectral variables was made in [5]. Many of the solutions obtained by the Baxterization formulae below belong to this list after the substitution $x=\exp(\mathrm{i}\theta)$ but some new solutions will appear as well.

We will now classify the constant triangular solutions of equation (2) that are of the form

$$R=\begin{pmatrix} q & 0 & 0 & 0 \\ x & r & 0 & 0 \\ g & y & s & 0 \\ p & h & z & t \end{pmatrix} \qquad \det R = qrst \neq 0. \tag{3}$$

Obviously there is a non-trivial intersection of the classes (1) and (3) so that many of the triangular solutions have appeared already in the classification performed in [2]. For those reasons, here we shall be interested mainly in solutions where $x\neq 0$ or $g\neq 0$ or $h\neq 0$ or $z\neq 0$.

Besides that, we shall exploit symmetries of the YBE

$$R\mapsto k(A\otimes A)R(A\otimes A)^{-1} \qquad A\in \mathrm{GL}(2,\mathbb{C}) \qquad k\in\mathbb{C}\backslash\{0\} \tag{4}$$

$$R\mapsto R^{\mathrm{T}}. \tag{5}$$

A very useful transformation will prove to be the transposition with respect to the antidiagonal, which is the composition of the above two

$$R\mapsto(\sigma\otimes\sigma)R^{\mathrm{T}}(\sigma\otimes\sigma)^{-1} \qquad \sigma=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6}$$

Let us rewrite (2) as

$$Y_{ijk}^{lmn} := R_{ij}^{ab}R_{ak}^{lc}R_{bc}^{mn} - R_{jk}^{bc}R_{ic}^{an}R_{ab}^{lm} = 0. \tag{7}$$

Inserting (3) into (7) we get 18 independent equations for 10 entries of (3). From Y_{121}^{111}, Y_{122}^{112}, Y_{221}^{211}, Y_{222}^{212} we can solve $r-q$ and $s-t$, or more precisely, exploiting also Y_{211}^{111} and Y_{222}^{122} we get the following possibilities:

$(\alpha 1)$	$g\neq 0$	$r-q=x(q-s-y)/g$	$s-t=z(q-s-y)/g$
$(\alpha 2)$	$g=0$	$q=s+y$ $\quad xy=0$	
$(\alpha 3)$	$g=0$	$q\neq s+y$ $\quad x=z=0$	
$(\beta 1)$	$h\neq 0$	$r-q=x(t-r-y)/h$	$s-t=-z(t-r-y)/g$
$(\beta 2)$	$h=0$	$t=r+y$ $\quad zy=0$	
$(\beta 3)$	$h=0$	$t\neq r+y$ $\quad x=z=0.$	

To proceed further we combine the possibilities (αi) and (βj). However, the cases $(\alpha 2,\beta 3)$, $(\alpha 3,\beta 2)$, $(\alpha 3,\beta 3)$ yield the solutions of the form (1) (with $c=d=0$) classified in [2]. Further, the cases $(\alpha 2,\beta 1)$ and $(\alpha 3,\beta 1)$ are transformable by the transformation (6) to $(\alpha 1,\beta 2)$ and $(\alpha 1,\beta 3)$. It means that only four combinations, namely $(\alpha 1,\beta 1)$, $(\alpha 2,\beta 1)$, $(\alpha 3,\beta 1)$ and $(\alpha 2,\beta 2)$, must be investigated.

Exploiting then the other equations (7) we get a list of triangular solutions to the YBE. Most of them can be transformed to the form (1.1) with $c=d=0$. For example, investigating the case $(\alpha 2,\beta 1)$ one finds a solution

$$R=\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & q+t & -t & 0 \\ p & h & -ht/q & t \end{pmatrix} \tag{8}$$

that can be transformed to the solution R_3 of the appendix. Nevertheless, there are solutions that cannot be transformed by (4) to the eight-or-less vertex form (1).

The result then is that up to the symmetries (4) and (5) there are just seven triangular solutions of the YBE. Five of them R_3, R_5, R_6, R_7, R_8 appeared in the classification of the eight-or-less vertex solutions. The remaining two are

$$R_{10}=\begin{pmatrix}1&0&0&0\\a&1&0&0\\b&0&1&0\\c&b&a&1\end{pmatrix}\qquad R_{11}=\begin{pmatrix}1&0&0&0\\-g&1&0&0\\g&0&1&0\\-gh&h&-h&1\end{pmatrix}. \tag{9}$$

The solution R_{11} with $g=-h=1$ appeared in [6].

Next we Baxterize the R-matrices that satisfy the (Hecke) condition

$$(PR-\lambda_1)(PR-\lambda_2)=0 \tag{10}$$

where P is the permutation matrix $P_{ij}^{kl}=\delta_j^k\delta_i^l$. In other words, the minimal polynomial of the matrix $\hat{R}:=PR$ is of the second degree. They are the matrices R_1-R_6 from the appendix and R_{11}.

It was proved [3] that if R is a constant solution of the YBE then

$$R(x)=R+\lambda_1\lambda_2 xPR^{-1}P \tag{11}$$

is a solution of the 'spectral-dependent' YBE

$$R_{12}(x)R_{13}(xy)R_{23}(y)=R_{23}(y)R_{13}(xy)R_{12}(x). \tag{12}$$

It turns out that if x is replaced by $\exp(i\theta)$ the solutions may be expressed in terms of the trigonometric functions.

Baxterization of the matrices R_1, R_2, R_5, R_6 yields solutions $8V(III)$, $8V(II)$, $6V(I)$, $6V(II)$ presented in [5] (for R_2 see also [7]).

The minimal polynomial of $\hat{R}_3$ is of the second degree

$$(\hat{R}_3-1)(\hat{R}_3+t)=0 \tag{13}$$

so that the eigenvalues are $\lambda_1=1$, $\lambda_2=-t$ and the Baxterization formula (11) gives the solution

$$R_3(x)=\begin{pmatrix}1-xt&0&0&0\\0&s(1-x)&x(1-t)&0\\0&1-t&st(1-x)&0\\(1-x)&0&0&x-t\end{pmatrix}. \tag{14}$$

The matrix R_4 gives $PR_3(x)P$.

Due to the fact that $PR_{11}^{-1}P=R_{11}$, the Baxterization formula (11) supplies only the non-essential factor $(-x)$ to the constant solution R_{11}.

There are unproved, but usually working, formulae for Baxterization such R-matrices that $\hat{R}$ have the minimal polynomial of degree three [3, 8]:

$$R(x)=a(x-1)R+bxP+cx(x-1)PR^{-1}P \tag{15}$$

where

$$a=-\lambda_3^{-1}\qquad b=1+\lambda_1\lambda_2^{-1}+\lambda_1\lambda_3^{-1}+\lambda_2\lambda_3^{-1}\qquad c=\lambda_1 \tag{16}$$

or

$$a=-\lambda_1\lambda_2^{-1}\lambda_3^{-1}\qquad b=1+\lambda_1\lambda_2^{-1}+\lambda_1\lambda_3^{-1}+\lambda_1^2\lambda_2^{-1}\lambda_3^{-1}\qquad c=\lambda_1 \tag{17}$$

where λ_1, λ_2, λ_3 are eigenvalues of $\hat{R}$. As the eigenvalues do not appear symmetrically in (16), (17) their permutations may yield different functions. In the cases below, there are always two eigenvalues with the opposite signs and then (16) and (17) give only three possibilities

$$\begin{aligned} &a=\varepsilon/p \qquad b=0 \qquad c=q \\ &a=-1/q \qquad b=0 \qquad c=\varepsilon p \\ &a=\varepsilon/p \qquad b=\varepsilon(p/q-q/p) \qquad c=\varepsilon p \end{aligned} \tag{18}$$

where $\varepsilon^2=1$ and $\pm p$, q are the eigenvalues of $\hat{R}$.

Using these formulae we can Baxterize constant solutions R_7, R_9 and special cases of R_8.

The roots of the minimal polynomial of $\hat{R}_7$ are

$$\lambda_1=-\lambda_2=\lambda_3=1. \tag{19}$$

All the solutions obtained from the formulae (15)-(17) are (up to a scalar factor) of the form

$$R_7(x)=\begin{pmatrix} 1+\varepsilon x & 0 & 0 & 0 \\ 0 & s(1+\varepsilon x) & 0 & 0 \\ 0 & 0 & s(1+\varepsilon x) & 0 \\ 1-\varepsilon x & 0 & 0 & 1+\varepsilon x \end{pmatrix} \qquad \varepsilon=\pm 1. \tag{20}$$

The matrix $\hat{R}_8$ has four eigenvalues in general. However, if $t=q$ or $t^2=rs$ or $q^2=rs$ then some of the eigenvalues coincide. For $t^2=q^2=rs$ we get the special cases of the matrices R_5 or R_6. For $q^2\neq rs$ and $t=q$ or $t^2=rs$, the minimal polynomial for $\hat{R}_8$ is of the third degree and

$$\lambda_1=-\lambda_2=\sqrt{rs} \qquad \lambda_3=q. \tag{21}$$

The Baxterization formulae (15), (18) then give three x-dependent R-matrices. Two of them are not very interesting as they are diagonal and any diagonal x-dependent matrix is a solution of the YBE. The third one given by

$$R(x)=p^{-1}(x-1)R_8+x(pq^{-1}-qp^{-1})+px(x-1)PR_8^{-1}P \qquad p^2=rs \tag{22}$$

is a solution of the YBE only if $t=q$. The trigonometric version of this solution is $6V(I)$ of [5]. If $t^2=rs\neq q^2$, the formula (22) does not yield a solution of the YBE.

The minimal polynomial for $\hat{R}_9$ is

$$(\hat{R}_9-1)(\hat{R}_9+1)(\hat{R}_9-t) \tag{23}$$

and the Baxterization formulae yield the solutions

$$R_{9a}(x)=\begin{pmatrix} 0 & 0 & 0 & 1\pm xt \\ 0 & 0 & t\pm x & 0 \\ 0 & t\pm x & 0 & 0 \\ 1\pm xt & 0 & 0 & 0 \end{pmatrix} \tag{24}$$

and

$$R_{9b}(x)=\begin{pmatrix} x(t-t^{-1}) & 0 & 0 & 1-x^2 \\ 0 & 0 & t-x^2t^{-1} & 0 \\ 0 & t-x^2t^{-1} & 0 & 0 \\ 1-x^2 & 0 & 0 & x(t-t^{-1}) \end{pmatrix}. \tag{25}$$

The matrices $\hat{R}_8$ (in the generic case) and $\hat{R}_{10}$ have minimal polynomials of the fourth degree. Then the Baxterization formula is

$$R(x) = A(x)RPR + B(x)R + C(x)P + D(X)PR^{-1}P \tag{26}$$

where A, B, C, D are quadratic and cubic functions of x with coefficients dependent in a rather complicated way on the eigenvalues of $\hat{R}$ [3]. The coefficients are not invariant with respect to the permutation of the eigenvalues so formula (26) represents several prescriptions for the Baxterization.

When applied to R_8, formula (26) gives either purely diagonal solutions or matrices that do not satisfy the YBE.

More interesting is the matrix R_{10}. The minimal polynomial for $\hat{R}_{10}$ is $(x-1)^3(x+1)$ so that the eigenvalues are ± 1. The Baxterization formula then gives four different solutions of the YBE. Two of them are constant up to a scalar factor and the other two are of the form

$$R_{10}(x) = \begin{pmatrix} q(x) & 0 & 0 & 0 \\ a(x) & q(x) & 0 & 0 \\ b(x) & 0 & q(x) & 0 \\ c(x) & b(x) & a(x) & q(x) \end{pmatrix} \tag{27}$$

where

$$\begin{aligned} &a(x) = a(2x+1) - bx(x+1) \qquad b(x) = b(2x+1) - ax(x+1) \\ &c(x) = c(1+x-x^2) - 2(a^2+b^2)x + 2abx^2 \qquad q(x) = x^2+3x+1 \end{aligned} \tag{28}$$

or

$$\begin{aligned} &a(x) = a - bx(x+1) \qquad b(x) = b - ax(x+1) \\ &c(x) = c(1-x-x^2) - 2(a^2+b^2)x + 2abx^2 \qquad q(x) = x^2+x+1. \end{aligned} \tag{29}$$

We have classified triangular 4×4 solutions of the YBE. In addition to R_3, R_5, R_6, R_7, R_8 (see appendix) obtained before, we have found two new constant solutions R_{10}, R_{11}.

We have applied the Baxterization formulae of [3] to the known constant solutions and obtained spectral-dependent solutions of the YBE (12) in this way. Several of them correspond to the trigonometric solutions obtained before in [5] but those displayed in this paper are new to the best knowledge of the author.

The calculations confirm that the Baxterization formulae for three and four eigenvalues of $\hat{R}$ are not universal but nevertheless may provide non-trivial solutions.

This work was supported in part by the grant CSAV 11 086.

Appendix: The eight-or-less vertex solutions of the YBE

$$R_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad R_1 = \begin{pmatrix} 1 & 0 & 0 & \mathrm{i} \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ \mathrm{i} & 0 & 0 & 1 \end{pmatrix}$$

$$R_2 = \begin{pmatrix} 1+t & 0 & 0 & 1 \\ 0 & \sqrt{1+t^2} & 1 & 0 \\ 0 & 1 & \sqrt{1+t^2} & 0 \\ 1 & 0 & 0 & 1-t \end{pmatrix}$$

$$R_3=\begin{pmatrix}1&0&0&0\\0&\pm1&0&0\\0&1-t&\pm t&0\\1&0&0&t\end{pmatrix}\qquad R_4=PR_3P$$

$$R_5=\begin{pmatrix}q&0&0&0\\0&1&0&0\\0&q-t&qt&0\\0&0&0&q\end{pmatrix}\qquad R_6=\begin{pmatrix}q&0&0&0\\0&1&0&0\\0&q-t&qt&0\\0&0&0&-t\end{pmatrix}$$

$$R_7=\begin{pmatrix}1&0&0&0\\0&\pm1&0&0\\0&0&\pm1&0\\1&0&0&1\end{pmatrix}\qquad R_8=\begin{pmatrix}q&0&0&0\\0&r&0&0\\0&0&s&0\\0&0&0&t\end{pmatrix}$$

$$R_9=\begin{pmatrix}0&0&0&1\\0&0&t&0\\0&t&0&0\\1&0&0&0\end{pmatrix}.$$

These matrices represent all regular solutions of the form (1) to the YBE (2) up to the symmetry

$$\tilde{R}=k(T\otimes T)R(T\otimes T)^{-1}\qquad k\in\mathbb{C}\qquad T\in\mathrm{GL}(2,\mathbb{C}).$$

Note added. After finishing this work I got to know about the preprint [9] where the solution R_{11} was independently found.

References

[1] Yang C N and Ge M-L (eds) 1989 *Braid Group, Knot Theory and Statistical Mechanics* (Singapore: World Scientific)
[2] Hlavatý L 1987 *J. Phys. A: Math. Gen.* **20** 1661
[3] Ge M-L, Wu Y S and Xue Y 1990 *Preprint* Stony Brook ITP-SB-90-02
Cheng Y, Ge M-L and Xue Y 1991 *Commun. Math. Phys.* **136** 195
[4] Akutsu Y and Deguchi T 1991 *Phys. Rev. Lett.* **67** 777
[5] Sogo K, Uchinami M, Akutsu Y and Wadati M 1982 *Prog. Theor. Phys.* **68** 508
[6] Demidov E E, Manin Yu I, Mukhin E E and Zhdanovich D V 1990 *Prog. Theor. Phys. Suppl.* **102** 203
[7] Ge M-L, Gwa L-H and Zhao H-K 1990 *J. Phys. A: Math. Gen.* **23** L795
[8] Couture M, Cheng Y, Ge M-L and Xue K 1991 *Int. J. Mod. Phys.* A **6** 559
[9] Ewen H, Ogievetski O and Wess J 1991 *Preprint* MPI-PAE/PTh 18/91